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Author(s)	Oshima, Yoichi
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ON AN OPTIMAL STOPPING PROBLEM OF TIME INHOMOGENEOUS DIFFUSION PROCESSES

YOICHI OSHIMA*

Abstract. For given quasi-continuous functions g, h with $g \leq h$ and diffusion process $\mathbf{M}$ determined by stochastic differential equations or symmetric Dirichlet forms, characterizations of the value functions $\widetilde{e}_g(s,x) = \sup_\sigma J_{(s,x)}(\sigma)$ and $\bar{w}(s,x) = \inf_\tau \sup_\sigma J_{(s,x)}(\sigma,\tau)$ are well studied so far. In this paper, by using the time dependent Dirichlet forms, we generalize these results to time inhomogeneous diffusion processes. The difficulty of our case arises from the existence of essential semipolar sets. In particular, excessive functions are not necessarily continuous along the sample paths. We get the result by showing such continuity of the value functions.

Key words. time-inhomogeneous diffusion processes, optimal stopping, Dirichlet forms, quasi-variational inequality

AMS subject classifications. 60G40, 49J40, 62L15,60J60

1. Introduction and Preliminaries. Let $\mathbf{M} = (X_t, P_{(s,x)})$ be a, not necessarily time homogeneous, diffusion process on a locally compact separable metric space X. For given (quasi-) continuous functions g, h on $[0,\infty) \times X$ and stopping times σ and τ, let

$$J_{(s,x)}(\sigma) = E_{(s,x)}\left(e^{-\sigma} g(s+\sigma, X_\sigma)\right), \tag{1.1}$$

$$J_{(s,x)}(\sigma,\tau) = E_{(s,x)}\left(e^{-\sigma\wedge\tau}\left(g(s+\sigma, X_\sigma) I_{\{\sigma\leq\tau\}} + h(s+\tau, X_\tau) I_{\{\tau<\sigma\}}\right)\right). \tag{1.2}$$

The main purpose of this paper is to characterize $\widetilde{e}_g(s,x) = \sup_\sigma J_{(s,x)}(\sigma)$ and $\bar{w}(s,x) = \sup_\sigma \inf_\tau J_{(s,x)}(\sigma,\tau)$.

Usually, such problem is considered for

$$J^f_{(s,x)}(\sigma,\tau) = E_{(s,x)}\left(\int_0^{\sigma\wedge\tau} e^{-t} f(s+t, X_t)\,dt\right) + J_{(s,x)}(\sigma,\tau) \tag{1.3}$$

instead of $J_{(s,x)}(\sigma,\tau)$. But we use $J_{(s,x)}(\sigma,\tau)$ because (1.3) is essentially reduced to (1.2) by taking $g + R_1 f$ and $h + R_1 f$ instead of g and h in (1.3), respectively, where $R_\alpha f$ is the resolvent of $\mathbf{M}$.

There are lot of works related to our problem. In particular, when $\mathbf{M}$ is a diffusion process determined by a stochastic differential equation with Lipschitz continuous coefficients, the detailed results related to $\widetilde{e}_g$ can be found in [1], [7] and references therein. In the time homogeneous case, Nagai [10], [11] and Zabczyk [19] used (symmetric) Dirichlet form theory to solve the problem. The diffusion process $\mathbf{M}$ corresponding to the generator on R^d determined by

$$A\varphi(x) = \sum_{i,j=1}^d \frac{1}{\rho(x)} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\rho(x)\frac{\partial \varphi}{\partial x_j}\right) \tag{1.4}$$

for a uniformly elliptic functions $(a_{ij}(x))_{i,j=1,2,\ldots,d}$ and a function $\rho(x) > 0$ belonging to a Sobolev space on R^d is contained in their framework. See also [5] and [8] for related results.

* Department of Mathematics, Faculty of Engineering, Kumamoto University, Kumamoto, 860-8555, Japan(oshima@gpo.kumamoto-u.ac.jp).

The purpose of this paper is to generalize those results to time inhomogeneous diffusion processes including the case that (a_{ij}) in (1.4) admits to depend on time parameter. In this case, the generator for each t is given by

$$A^{(t)}\varphi(x) = \sum_{i,j=1}^{d} \frac{1}{\rho(x)} \frac{\partial}{\partial x_i} \left(a_{ij}(t,x)\rho(x) \frac{\partial \varphi}{\partial x_j} \right) \tag{1.5}$$

and the cooresponding Dirichlet form on $L^2(R^d; \rho(x)dx)$ is an extension of

$$E^{(t)}(\varphi, \psi) = \sum_{i,j=1}^{d} \int_{R^d} a_{ij}(t,x) \frac{\partial \varphi}{\partial x_i} \frac{\partial \psi}{\partial x_j} \rho(x)dx. \tag{1.6}$$

In the Lipschitz continuous and time homogeneous cases stated above, the (quasi-) continuity of the value functions $\widetilde{e}_g$ and $\bar{w}$ follows naturally. The essential step in this paper is to prove the fine and cofine continuities of the value functions.

The organization of this paper is as follows. In the rest of this section, the notions of time dependent Dirichlet forms and the basic properties of the associated time inhomogeneous Markov processes are stated. In section 2, under the separability condition, quasi-variational inequalities and their solutions are given. In section 3, the optimal stopping problem is solved dividing into three cases; (I) one obstacle cases, (II) two obstacles cases under the separability condition and (III) general two obstacles cases.

Now we shall start with our settings. Let X be a locally compact separable metric space and m a positive Radon measure on X with full support. We assume that we are given a family $(E^{(t)}, F)_{t \geq 0}$ of Dirichlet forms on $H = L^2(X; m)$ satisfying the following conditions:
(i) For each $t \geq 0$, $(E^{(t)}, F)$ is an m-symmetric Dirichlet form on H.
(ii) For any $\varphi \in F$, $E^{(t)}(\varphi, \varphi)$ is measurable function of $t \geq 0$ and satisfies

$$\lambda^{-1} \|\psi\|_F^2 \leq E_1^{(t)}(\psi, \psi) \leq \lambda \|\psi\|_F^2, \tag{1.7}$$

for some positive constant λ, where $E_\alpha^{(t)}(\psi, \psi) = E^{(t)}(\psi, \psi) + \alpha(\psi, \psi)_m$ and $\|\psi\|_F^2 = E_1^{(0)}(\psi, \psi)$.
(iii) F is regular, that is $C_0(X) \cap F$ is uniformly dense in $C_0(X)$ and $\|\cdot\|_F$-dense in F, where $C_0(X)$ is the family of continuous functions on X with compact support.
(iv) For any $t \geq 0$, $(E^{(t)}, F)$ is local, that is, for any $\varphi, \psi \in F \cap C_0(X)$ such that $\varphi \cdot \psi = 0$, $E^{(t)}(\varphi, \psi) = 0$.

For simplicity, we put $E^{(t)} = E^{(0)}$ for $t < 0$. For each t, there exists an operator $A^{(t)}$ from F to F' such that

$$-\langle A^{(t)}\varphi, \psi \rangle = E^{(t)}(\varphi, \psi), \tag{1.8}$$

for any $\varphi, \psi \in F$. To consider an optimal stopping problem related to the time inhomogeneous diffusion process X_t with generator $A^{(t)}$, we shall introduce the space-time process $Z_t = (\tau(t), X_t)$ on $Z = R^1 \times X$ with uniform motion $\tau(t)$. Formally, the resolvent $R_\alpha f$ of Z_t satisfies

$$\left(\alpha - \frac{\partial}{\partial t} - A^{(t)} \right) R_\alpha f(t,x) = f(t,x). \tag{1.9}$$

To define Z_t more rigorously, let us introduce the spaces $\mathcal{H}$, $\mathcal{F}$ and $\mathcal{W}$. Put $\mathcal{H} = \{u(t,x) : u(t,\cdot) \in H, \|u\|_{\mathcal{H}} < \infty\}$, where

$$\|u\|_{\mathcal{H}}^2 = \int_{R^1} \|u(t,\cdot)\|_H^2 dt.$$

The space $\mathcal{F}$ is a family of measurable functions $u \in \mathcal{H}$ such that $u(t,\cdot) \in F$ for all t and $\|u\|_{\mathcal{F}} < \infty$, where

$$\|u\|_{\mathcal{F}}^2 = \int_{R^1} \|u(t,\cdot)\|_F^2 \, dt.$$

The dual space $\mathcal{F}'$ is defined similarly by taking F' instead of F in the definition of $\mathcal{F}$. For any function $f \in \mathcal{F}$, considering f as function of $t \in R^1$ with value in F', the distribution sense derivative $\partial f/\partial t$ is defined as a function $g(t,\cdot)$ on R^1 with value in F' such that

$$\int_{R^1} g(t,\cdot)\xi(t)dt = \int f(t,\cdot)\xi'(t)dt$$

for any $\xi \in C_0^\infty(R^1)$. Using this derivative, define the space $(\mathcal{W}, \|\cdot\|_{\mathcal{W}})$ by

$$\mathcal{W} = \left\{ u \in \mathcal{F} : \frac{\partial u}{\partial t} \in \mathcal{F}', \ \|u\|_{\mathcal{W}} < \infty \right\}$$
$$\|u\|_{\mathcal{W}}^2 = \left\| \frac{\partial u}{\partial t} \right\|_{\mathcal{F}'}^2 + \|u\|_{\mathcal{F}}^2 .$$

Further define the bilinear form $\mathcal{E}$ by

$$\mathcal{E}(u,v) = \begin{cases} -\int_{R^1} \left(\frac{\partial u}{\partial t}, v \right) dt + \int_{R^1} E^{(t)}(u(t,\cdot), v(t,\cdot))dt, & u \in \mathcal{W}, v \in \mathcal{F} \\ \int_{R^1} \left(\frac{\partial v}{\partial t}, u \right) dt + \int_{R^1} E^{(t)}(u(t,\cdot), v(t,\cdot))dt, & u \in \mathcal{F}, v \in \mathcal{W}. \end{cases}$$

Then, for $f \in \mathcal{H}$, $R_\alpha f$ in (1.9) is considered as a version of $G_\alpha f \in \mathcal{W}$ of the solution of

$$\mathcal{E}_\alpha(G_\alpha f, v) = (f, v)_\nu,$$

for any $v \in \mathcal{F}$, where $\mathcal{E}_\alpha(\ ,\ ) = \mathcal{E}(\ ,\ ) + \alpha(\ ,\ )_\nu$ and $d\nu(t,x) = dtdm(x)$. This equation is equivalent to

$$-\left(\frac{\partial}{\partial t} G_\alpha f(t,\cdot), \varphi \right) + E_\alpha^{(t)} \left(G_\alpha f(t,\cdot), \varphi \right) = (f(t,\cdot), \varphi)$$

for any $t \geq 0$ and $\varphi \in F$. The dual resolvent $\widehat{G}_\alpha f \in \mathcal{W}$ is defined as a solution of

$$\left(\frac{\partial}{\partial t} \widehat{G}_\alpha f(t,\cdot), \varphi \right) + E_\alpha^{(t)} \left(\varphi, \widehat{G}_\alpha f(t,\cdot) \right) = (f(t,\cdot), \varphi).$$

for any $t \geq 0$ and $\varphi \in F$. Then, for any $f \in \mathcal{F}$ [resp. $f \in \mathcal{H}$], $\|\alpha G_\alpha f\|_{\mathcal{F}} \leq C_1 \|f\|_{\mathcal{F}}$ [resp. $\|\alpha G_\alpha f\| \leq \|f\|_{\mathcal{H}}$] for some constant C_1 and $\lim_{\alpha\to\infty} \alpha G_\alpha f = f$ in $\mathcal{F}$ [resp. $\mathcal{H}$]

(see [[14];Lemma 2.1], [[16];I.3, I.4]). Similar results also hold for the dual resolvent $\widehat{G}_\alpha$.

To choose a version $R_\alpha f$ of $G_\alpha f$, we need to define a capacity. A function $u \in \mathcal{F}$ is called *α-excessive* if $\mathcal{E}_\alpha(u,w) \geq 0$ for any non-negative function $w \in \mathcal{W}$. Then $u \in \mathcal{F}$ is α-excessive if and only if $u \geq 0$ and $\beta G_{\beta+\alpha} u \leq u$ a.e. for all $\beta > 0$ (see [14]). We denote by $\mathcal{P}_\alpha$ the family of all α-excessive functions. In particular put $\mathcal{P} = \mathcal{P}_1$.

For any function $h \in \mathcal{H}$, let

$$\mathcal{L}_h = \{u \in \mathcal{F} : u \geq h \ \nu\text{-}a.e.\}$$

and $\mathcal{L}_A = \mathcal{L}_{I_A}$. Then the following results hold (see [9] and [15]).

Lemma 1.1. *For any $\varepsilon > 0$ and $\alpha > 0$, there exists a unique function $h_\varepsilon^\alpha \in \mathcal{W}$ such that*

$$-\left(\frac{\partial h_\varepsilon^\alpha}{\partial t}, \varphi\right) + E_\alpha^{(t)}\left(h_\varepsilon^\alpha(t,\cdot), \varphi\right) = \frac{1}{\varepsilon}\left(\left(h_\varepsilon^\alpha(t,\cdot) - h(t,\cdot)\right)^-, \varphi\right) \tag{1.10}$$

for any $\varphi \in F$.

Theorem 1.2. *Suppose that $\mathcal{L}_h \cap \mathcal{W} \neq \emptyset$. Then $e_h^\alpha = \lim_{\varepsilon \downarrow 0} h_\varepsilon^\alpha$ converges increasingly, strongly in $\mathcal{H}$ and weakly in $\mathcal{F}$. Furthermore, e_h^α is the minimal function of $\mathcal{P}_\alpha \cap \mathcal{L}_h$ and satisfies*

$$\mathcal{A}_\alpha\left(e_h^\alpha, e_h^\alpha\right) \leq \mathcal{E}_\alpha\left(e_h^\alpha, w\right), \tag{1.11}$$

for any $w \in \mathcal{L}_h \cap \mathcal{W}$.

If $u \in \mathcal{P}_\alpha$, then there exists a positive Radon measure μ_u^α on Z such that

$$\mathcal{E}_\alpha(u,w) = \int_Z w(z) d\mu_u^\alpha(z), \quad \text{for any } w \in C_0(Z) \cap \mathcal{W}. \tag{1.12}$$

We omit the superfix α in e_h^α and μ_u^α if $\alpha = 1$. For any open set A of Z such that $\mathcal{L}_A \cap \mathcal{W} \neq \emptyset$, put $e_A = e_{I_A}$ and $\mu_A = \mu_{e_A}$. Then μ_A is supported by the closure $\bar{A}$ of A. The *capacity* $\mathrm{Cap}(A)$ of A is defined by

$$\mathrm{Cap}(A) = \mu_A(\bar{A}).$$

If there exists $w \in \mathcal{W}$ such that $w = 1$ a.e. on A, then

$$\mathrm{Cap}(A) = \mathcal{E}_\alpha(e_A^\alpha, w). \tag{1.13}$$

The notion of the capacity is extended to any Borel set by the usual manner. A set is called *exceptional* if it is of zero capacity. If a statement holds except on an exceptional set, then it is called that the statement holds quasi-everywhere (q.e. in abbreviation).

An increasing sequence of closed sets $\{F_n\}$ is called a *nest* if $\lim_{n\to\infty} \mathrm{Cap}(Z \backslash F_n) = 0$. A function u is called *quasi-continuous* (q.c. in abbreviation) if, there exists a nest $\{F_n\}$ of closed sets such that u is continuous on each F_n. The *quasi-lower semi-continuity* is defined similarly. Any function $u \in \mathcal{W}$ has a q.c. modification $\widetilde{u}$. In particular, for any $f \in \mathcal{H}$ and $\alpha > 0$, $G_\alpha f$ and $\widehat{G}_\alpha f$ have quasi-continuous modifications. The relation (1.12) can be extended to $w \in \mathcal{W}$ by taking the q.c. modification. For any α-excessive function $u \in \mathcal{F}$, define its α-excessive modification $\widetilde{u}$ by $\widetilde{u} = \lim_{n\to\infty} nR_{n+\alpha} u$. Since $\widetilde{u}$ is an increasing limit of quasi-continuous functions,

$\widetilde{u}$ is quasi-lower semi-continuous. The following theorem and the properties of the associated diffusion process can be found in [12], [14], [16], [17] and [18].

THEOREM 1.3. *There exist diffusion processes* $\mathbf{M} = (Z_t, P_z)$ *and* $\widehat{\mathbf{M}} = (\widehat{Z}_t, \widehat{P}_z)$ *on* Z *satisfying the following conditions.*
(i) The resolvents $R_\alpha f$ *and* $\widehat{R}_\alpha f$ *of* $\mathbf{M}$ *and* $\widehat{\mathbf{M}}$ *are quasi-continuous modifications of* $G_\alpha f$ *and* $\widehat{G}_\alpha f$*, respectively.*
(ii) Let $Z_t = (\tau(t), X_t)$ *and* $\widehat{Z}_t = (\widehat{\tau}(t), \widehat{X}_t)$ *be the decompositions of* Z_t *and* $\widehat{Z}_t$ *into the processes on* R^1 *and* X *respectively. Then* $\tau(t) = \tau(0) + t$ *and* $\widehat{\tau}(t) = \widehat{\tau}(0) - t$*.*
(iii) For any open set A *of* Z*,* $E.(e^{-\alpha\sigma_A})$ *is a quasi-lower semi-continuous modification of* e_A^α*, where* σ_A *is the hitting time of* A*.*

For later use, we present two lemmas. The proof of Lemma 1.4 can be found in [[12];Lemma 3.7].

LEMMA 1.4. *For any* α*-excessive function* $u \in \mathcal{F}$*,* $\mu_u^{(\alpha)}$ *does not charge any Borel set of zero capacity.*

LEMMA 1.5. *Suppose that a sequence of* 1*-excessive functions* $\{u_n\}$ *converges to zero in* $\mathcal{H}$*. Then, there exists a subsequence* $\{u_{n_k}\}$ *such that* $\lim_{k\to\infty} \widetilde{u}_{n_k} = 0$ *quasi-uniformly, that is there exists a nest* $\{F_n\}$ *such that* $\lim_{k\to\infty} \widetilde{u}_{n_k} = 0$ *uniformly on each* F_n*.*

Proof. Since $\widetilde{u}_n$ is quasi-lower semi-continuous, there exists an open set N_k such that $\mathrm{Cap}(N_k) < 1/2^k$ and $\widetilde{u}_n$ is lower semi-continuous on $Z \setminus N_k$ for all n. Put $B_k^n = \{z \in Z \setminus N_k : \widetilde{u}_n(z) > 1/2^k\}$ and $D_k^n = B_k^n \cup N_k$. Then D_k^n is open and, noting that $(e_{N_k}, p) = \langle \mu_{N_k}, \widehat{R}_1 p\rangle \le \|p\|_\infty \mathrm{Cap}(N_k) < 1/2^k$ for any non-negative bounded continuous function $p \in \mathcal{H}$, it holds that

$$\begin{aligned}(e_{D_k^n}, p) &\le (e_{B_k^n}, p) + (e_{N_k}, p) \le 2^k\,(u_n, p) + \|p\|_\infty \mathrm{Cap}(N_k)\\ &\le 2^k \|u_n\|_{\mathcal{H}} \cdot \|p\|_{\mathcal{H}} + \frac{\|p\|_\infty}{2^k}.\end{aligned}$$

For each k, take n_k such that $\|u_n\|_{\mathcal{H}} \le 1/2^{2k}$ for any $n \ge n_k$. Then $F_m = Z \setminus \cup_{k=m}^\infty D_{n_k}^k$ is a closed set. Since $\{Z \setminus F_m\}$ is a decreasing sequence of open sets such that

$$\langle \mu_{Z\setminus F_m}, \widehat{R}_1 p\rangle = \left(e_{Z\setminus F_m}, p\right) \le \frac{1}{2^{m-1}}\left(\|p\|_{\mathcal{H}} + \|p\|_\infty\right),$$

for any p satisfying the stated conditions, $\lim_{m\to\infty} \mathrm{Cap}(Z\setminus F_m) = \lim_{m\to\infty} \mu_{Z\setminus F_m}(Z) = 0$. Furthermore, $\lim_{k\to\infty} u_{n_k} = 0$ uniformly on each F_m. □

2. Quasi-variational inequalities. In this section, we assume that we are given two *obstacles* $g, h \in \mathcal{F}$ which are quasi-continuous and $g \le h$ q.e. We say that the pair (g, h) satisfies the *separability condition* if there exist $\varphi, \psi \in \mathcal{P}$ such that

$$g \le \widetilde{\varphi} - \widetilde{\psi} \le h \quad q.e. \tag{2.1}$$

Define the sequences of 1-excessive functions $\{u_n\}$ and $\{v_n\}$ inductively by

$$u_0 = 0,\ v_n = e_{u_{n-1}-h},\ u_n = e_{v_n+g}.$$

For any $\phi \in \mathcal{P}$, let L_ϕ be a continuous linear functional on $\mathcal{F}$ defined by $L_\phi(w) = \mathcal{A}_1(\phi, w)$. In the proof of Lemma 5.2 in [14], one can see that $2\widehat{G}_1 L_\phi - \phi$ is 1-coexcessive, thus in particular non-negative. Hence

$$\phi \le 2\widehat{G}_1 L_\phi \in \mathcal{W}. \tag{2.2}$$

LEMMA 2.1. *Suppose that the separability condition (2.1) holds. Then*
(i) u_n, v_n *are well defined.*
(ii) $\lim_{n\to\infty} u_n = \bar{u}$ *and* $\lim_{n\to\infty} v_n = \bar{v}$ *converge increasingly, strongly in* $\mathcal{H}$ *and weakly in* $\mathcal{F}$.
(iii) $\bar{u} \le \varphi$, $\bar{v} \le \psi$ *and* $g \le \bar{u} - \bar{v} \le h$ *a.e.*

Proof. Clearly $u_0 = 0 \le \varphi$. Suppose that u_{n-1} is defined and satisfies $u_{n-1} \le \varphi$. Then by the separability condition, $u_{n-1} - h \le \varphi - h \le \psi$. Thus $u_{n-1} - h \le 2\widehat{G}_1 L_\psi \in \mathcal{W}$ by (2.2). Hence $v_n := e_{u_{n-1}-h}$ is well defined and $v_n \le \psi$ by Theorem 1.2. Again, by (2.2) and the separability condition, $v_n + g = e_{u_{n-1}-h} + g \le \psi + g \le \varphi \le 2\widehat{G}_1 L_\varphi \in \mathcal{W}$. Therefore $u_n := e_{v_n+g}$ is well defined and dominated by φ.
If $u_{n-1} \le u_n$, then $v_n = e_{u_{n-1}-h} \le e_{u_n-h} = v_{n+1}$ and $u_n = e_{v_n+g} \le e_{v_{n+1}+g} = u_{n+1}$. Thus u_n and v_n are well defined and increasing relative to n. By virtue of (2.2),

$$\mathcal{A}_1(u_n.u_n) \le 2\mathcal{E}_1(u_n, \widehat{G}_1 L_\varphi) \le 2\|u_n\|_{\mathcal{F}} \|\widehat{G}_1 L_\varphi\|_{\mathcal{W}}.$$

Hence $\{\|u_n\|_{\mathcal{F}}\}$ is bounded. Similarly, $\{\|v_n\|_{\mathcal{F}}\}$ is bounded and the assertion (ii) holds by Lemma I.2.12 in [6].
Since $u_n \le \varphi$ and $v_n \le \psi$, the first assertion of (iii) holds. Furthermore, from the definition, $u_{n-1} - h \le v_n$ and $v_n + g \le u_n$. This implies the second assertion of (iii).
□

THEOREM 2.2. *Under the separability condition,* $\bar{u} = e_{\bar{v}+g}$ *and* $\bar{v} = e_{\bar{u}-h}$. *In particular,*

$$\begin{aligned}\mathcal{A}_1(\bar{u}, \bar{u}) &\le \mathcal{E}_1(\bar{u}, w), \quad \forall w \in \mathcal{L}_{\bar{v}+g} \cap \mathcal{W},\\ \mathcal{A}_1(\bar{v}, \bar{v}) &\le \mathcal{E}_1(\bar{v}, w), \quad \forall w \in \mathcal{L}_{\bar{u}-h} \cap \mathcal{W}.\end{aligned}$$

Moreover, if a pair of 1*-excessive functions* (u, v) *satisfies* $g \le u - v \le h$, *then* $\bar{u} \le u$ *and* $\bar{v} \le v$.

Proof. Since $\bar{u}$ is a 1-excessive function in $\mathcal{L}_{\bar{v}+g}$, clearly $e_{\bar{v}+g} \le \bar{u}$. Conversely, $\bar{u} = \lim_{n\to\infty} u_n = \lim_{n\to\infty} e_{v_n+g} \le e_{\bar{v}+g}$ by Lemma 2.1. Similarly, $\bar{v} = e_{\bar{u}-h}$. The quasi-variational inequalities are already stated in Theorem 1.2. By Lemma 2.1, if g, h satisfies the separability condition with $(u, v) \in \mathcal{P} \times \mathcal{P}$, then $u_n \le u, v_n \le v$ for any n. Since $\lim_{n\to\infty} u_n = \bar{u}, \lim_{n\to\infty} v_n = \bar{v}$, we obtain $\bar{u} \le u, \bar{v} \le v$. □

Similar quasi-variational inequality for $\bar{u} - \bar{v}$ also holds. But it will be given in the next section because we use a probabilistic argument for the proof.

LEMMA 2.3. *For any* $g \in \mathcal{W}$, $\lim_{k\to\infty} e_{g-g^{(k)}} = 0$ *in* $\mathcal{F}$.

Proof. Since $g - g^{(k)} \in \mathcal{L}_{g-g^{(k)}} \cap \mathcal{W}$,

$$\begin{aligned}\mathcal{A}_1\left(e_{g-g^{(k)}}, e_{g-g^{(k)}}\right) &\le \mathcal{E}_1\left(e_{g-g^{(k)}}, g - g^{(k)}\right)\\ &= -\mathcal{E}_1\left(g - g^{(k)}, e_{g-g^{(k)}}\right) + 2\mathcal{A}_1\left(e_{g-g^{(k)}}, g - g^{(k)}\right)\\ &= -\mathcal{E}_1\left(g, e_{g-g^{(k)}} - k\widehat{G}_k e_{g-g^{(k)}}\right) + 2\mathcal{A}_1\left(e_{g-g^{(k)}}, g - g^{(k)}\right)\\ &\le 2\|g\|_{\mathcal{W}} \|e_{g-g^{(k)}} - k\widehat{G}_k e_{g-g^{(k)}}\|_{\mathcal{F}} + 2\|g - g^{(k)}\|_{\mathcal{F}} \|e_{g-g^{(k)}}\|_{\mathcal{F}}\\ &\le 4\|g\|_{\mathcal{W}}(C_1 + 1)\|e_{g-g^{(k)}}\|_{\mathcal{F}}.\end{aligned} \tag{2.3}$$

Hence $\mathcal{A}_1(e_{g-g^{(k)}}, e_{g-g^{(k)}})$ is bounded. By virtue of [[16];III.Lemma 2.2], $e_{g-g^{(k)}}$ converges to 0 strongly in $\mathcal{H}$ and hence weakly in $\mathcal{F}$ from [[6];Lemma I.2.12]. Since $\|k\widehat{G}_k e_{g-g^{(k)}}\|_{\mathcal{F}} \le C_1 \|e_{g-g^{(k)}}\|_{\mathcal{F}}$, by the same argument, $\lim_{k\to\infty} k\widehat{G}_k e_{g-g^{(k)}} = 0$ strongly in $\mathcal{H}$ and weakly in $\mathcal{F}$. Hence, from (2.3), $\lim_{k\to\infty} \mathcal{A}_1(e_{g-g^{(k)}}, e_{g-g^{(k)}}) = 0$.
□

3. An optimal stopping problem. Let $\mathbf{M}$ and $\widehat{\mathbf{M}}$ be the diffusion processes given by Theorem1.3. Denote by R_α and $\widehat{R}_\alpha$ their associated resolvents. For any stopping time σ, define $H_\sigma u$ by $H_\sigma u(z) = E_z\left(e^{-\sigma}u(Z_\sigma)\right)$. In particular, put $H_B = H_{\sigma_B}$ for the hitting time σ_B of the nearly Borel set B.

(I) One obstacle case:
Let g be a quasi-continuous function of $\mathcal{F}$ such that $\mathcal{L}_g \cap \mathcal{W} \neq \emptyset$. As in the previous section, denote by e_g the minimal 1-excessive function of $\mathcal{L}_g$. Then it is the minimal function of $\mathcal{L}_g$ satisfying the quasi-variational inequality (1.11). The following result is a time inhomogeneous version of Nagai's result [10].

THEOREM 3.1. *Suppose that $g \in \mathcal{F}$ is quasi-continuous and $\mathcal{L}_g \cap \mathcal{W} \neq \emptyset$. Then*

$$\widetilde{e}_g(z) = \sup_\sigma J_z(\sigma) = E_z\left(e^{-\sigma_B}g(Z_{\sigma_B})\right) \quad q.e., \tag{3.1}$$

where the supremum is taken over all stopping times σ and $B = \{z : \widetilde{e}_g(z) = g(z)\}$.

Proof. Noting that $\widetilde{e}_g$ is the smallest 1-excessive function dominating g q.e., we have for any stopping time σ,

$$E_z\left(e^{-\sigma}g(Z_\sigma)\right) \leq E_z\left(e^{-\sigma}\widetilde{e}_g(Z_\sigma)\right) \leq \widetilde{e}_g(z) \;\; q.e.$$

Hence it is enough to show

$$\widetilde{e}_g(z) = E_z\left(e^{-\sigma_B}g(Z_{\sigma_B})\right). \tag{3.2}$$

This is essentially shown in [[14], Lemma 6.2], but we shall give the outline of the proof for the completeness. For $\varepsilon_n \downarrow 0$, let g_n be a q.c. version of the solution of $g_n = (1/\varepsilon_n)G_1\left((g_n - g)^-\right)$ determined by Lemma1.1. Let $B_n = \{z : g_n(z) \leq g(z)\}$ and $\dot{\sigma}_n = \dot{\sigma}_{B_n}$, where $\dot{\sigma}_A$ is the first entry time of A defined by $\dot{\sigma}_A = \inf\{t \geq 0 : Z_t \in A\}$. Then

$$\begin{aligned} g_n(z) &= \frac{1}{\varepsilon_n}E_z\left(\int_{\dot{\sigma}_n}^\infty e^{-t}(g_n - g)^-(Z_t)dt\right) \\ &= E_z\left(e^{-\dot{\sigma}_n}g_n(Z_{\dot{\sigma}_n})\right) \leq E_z\left(e^{-\dot{\sigma}_n}g(Z_{\dot{\sigma}_n})\right) \\ &\leq E_z\left(e^{-\dot{\sigma}_n}\widetilde{e}_g(Z_{\dot{\sigma}_n})\right). \end{aligned}$$

Put $\dot{\sigma} = \lim_{n\to\infty}\dot{\sigma}_n$. Then $\dot{\sigma} \leq \dot{\sigma}_B$. By virtue of Theorem1.2, since $g_n \uparrow e_g$ a.e., we then have, for any non-negative function $f \in \mathcal{H}$,

$$\begin{aligned} (f, \widetilde{e}_g) &= \lim_{n\to\infty}(f, g_n) = \lim_{n\to\infty}E_{f\cdot\nu}\left(e^{-\dot{\sigma}_n}g_n(Z_{\dot{\sigma}_n})\right) \leq \lim_{n\to\infty}E_{f\cdot\nu}\left(e^{-\dot{\sigma}_n}g(Z_{\dot{\sigma}_n})\right) \\ &= E_{f\cdot\nu}\left(e^{-\dot{\sigma}}g(Z_{\dot{\sigma}})\right) \leq E_{f\cdot\nu}\left(e^{-\dot{\sigma}}\widetilde{e}_g(Z_{\dot{\sigma}})\right) \leq (f, \widetilde{e}_g). \end{aligned}$$

Hence $\widetilde{e}_g(z) = E_z\left(e^{-\dot{\sigma}}g(Z_{\dot{\sigma}})\right)$ for a.e.z. Since $g \leq \widetilde{e}_g$ q.e., we also have $g(Z_{\dot{\sigma}}) = \widetilde{e}_g(Z_{\dot{\sigma}})$ and hence $\dot{\sigma}_B \leq \dot{\sigma}$ a.s. P_z for a.e. z. Therefore $\dot{\sigma} = \dot{\sigma}_B$ a.s. P_z and $\widetilde{e}_g(z) = E_z\left(e^{-\dot{\sigma}_B}g(Z_{\dot{\sigma}_B})\right)$ for a.e. z. By taking the 1-excessive regularization, we get the result. □

REMARK: Since $\dot{\sigma}_B \leq \sigma_B$, it holds that

$$\widetilde{e}_g \geq H_{\dot{\sigma}_B}\widetilde{e}_g \geq H_B\widetilde{e}_g \geq H_B g \quad q.e.$$

Hence Theorem 3.1 implies that $H_{\dot{\sigma}_B}\widetilde{e}_g = H_B\widetilde{e}_g$ q.e.z. In particular, the set of irregular points of B is exceptional.

(II) Two obstacles case under the separability condition:
We assume that we are given two quasi-continuous functions $g, h \in \mathcal{F}$ such that $g \leq h$ q.e. If the separability condition (2.1) is satisfied, then there exists the minimal pair of finely continuous functions $(\bar{u}, \bar{v})$ given by Theorem 2.2. They are given by $\bar{u} = \lim_{k\to\infty} \bar{u}_k$ and $\bar{v} = \lim_{k\to\infty} \bar{v}_k$ with the solutions $\bar{u}_k, \bar{v}_k \in \mathcal{W}$ of $\bar{u}_k = (1/\varepsilon_k)G_\alpha\left((\bar{u}_k - \bar{v} - g)^-\right)$ and $\bar{v}_k = (1/\varepsilon_k)G_\alpha\left((\bar{v}_k - \bar{u} + h)^-\right)$, respectively. Define the sets C_k and D_k by

$$C_k = \{z : \bar{u}_k(z) \leq (g + \bar{v})(z)\}, \quad D_k = \{z : \bar{v}_k(z) \leq (\bar{u} - h)(z)\}. \tag{3.3}$$

Then $C_k \supset C$ and $D_k \supset D$ for

$$C = \{z : \bar{u}(z) = (g + \bar{v})(z)\}, \quad D = \{z : \bar{v}(z) = (\bar{u} - h)(z)\}. \tag{3.4}$$

As we remarked before (2.2), $\bar{u}$ can be represented as a difference of a function of $\mathcal{W}$ and a co-excessive function. In particular, $\bar{u}$ has a q.e. cofinely continuous modification $\widehat{u}$ given by $\widehat{u} = \lim_{n\to\infty} n\widehat{R}_{n+1}\bar{u}$. Since $\bar{u}$ is q.e. lower-semicontinuous, $\widehat{u}(z) = \lim_{t\to 0} \widehat{E}_z\left(\bar{u}(Z_t)\right) \geq \underline{\lim}_{y\to z} \bar{u}(y) \geq \bar{u}(z)$ q.e. Similarly a q.e. cofinely continuous modification $\widehat{v}$ of $\bar{v}$ exists and satisfies $\widehat{v} \geq \bar{v}$ q.e.

Since $\bar{v}$ is 1-excessive, there exists a positive Radon measure $\mu_{\bar{v}}$ charging no exceptional set such that $\mathcal{E}_1(\bar{v}, w) = \langle \mu_{\bar{v}}, \widetilde{w}\rangle$ for any $w \in \mathcal{W}$. In the following two lemmas, we use the notation $\mu_{\bar{v},F} = \mu_{\bar{v}}|_F, \mu_{\bar{v},F^c} = \mu_{\bar{v}}|_{F^c}, \bar{v}_F = \widetilde{U}_1\mu_{\bar{v},F}$, $\bar{v}_{F^c} = \widetilde{U}_1\mu_{\bar{v},F^c}$ and $\widehat{v}_{F^c} = \lim_{n\to\infty} n\widehat{R}_{n+1}\bar{v}_{F^c}$.

Lemma 3.2. *Assume that there exists a non-exceptional compact set F such that $F \subset \{z : (\widehat{v} - \bar{v})(z) \geq \delta\}$ for some $\delta > 0$. Then $\widehat{v}_{F^c}(z) = \bar{v}_{F^c}(z)$ for q.e. $z \in F$.*

Proof. For the simplicity of the notation, put $\mu_1 = \mu_{\bar{v},F}$, $\mu_2 = \mu_{\bar{v},F^c}$, $\bar{v}_2 = \widetilde{U}_1\mu_2$ and $\widehat{v}_2$ the cofinely continuous modification of $\bar{v}_2$. Assume that there exists a non-exceptional compact subset K of F such that $\bar{v}_2(z) < \widehat{v}_2(z)$ for q.e. $z \in K$. For a decreasing sequence of open sets G_n such that $\bar{G}_{n+1} \subset G_n$ and $\cap_n G_n = K$, since σ_{G_n} increases strictly to σ_K a.s.P_z for q.e. $z \notin K$, the left continuity of $\widehat{v}_2(Z_t)$ implies that

$$\lim_{n\to\infty} H_{G_n}\bar{v}_2(z) = \lim_{n\to\infty} E_z\left(e^{-\sigma_{G_n}}\bar{v}_2(Z_{\sigma_{G_n}}\right) = E_z\left(e^{-\sigma_K}\widehat{v}_2(Z_{\sigma_K})\right) = H_K\widehat{v}_2(z).$$

On the other hand, since $\mu_2(K) = 0$, for any $f \in L^2_+(Z)$, we have from [[14]; Corollary 5.1] and [[2]; Theorem I.11.2].

$$\begin{aligned}
\lim_{n\to\infty}(H_{G_n}\bar{v}_2, f) &= \lim_{n\to\infty}\mathcal{E}_1(H_{G_n}\bar{v}_2, \widehat{R}_1 f) = \lim_{n\to\infty}\mathcal{E}_1(U_1\mu_2, \widehat{H}_{G_n}\widehat{R}_1 f)\\
&= \lim_{n\to\infty}\langle \mu_2, \widehat{H}_{G_n}\widehat{R}_1 f\rangle = \langle \mu_2, \widehat{H}_K\widehat{R}_1 f\rangle\\
&= (H_K\bar{v}_2, f).
\end{aligned}$$

Hence $H_K\widehat{v}_2 = H_K\bar{v}_2$ a.e. which contradicts to the assumption. □

Lemma 3.3. *$\widehat{u} = \bar{u}$ and $\widehat{v} = \bar{v}$ q.e.*

Proof. We shall divide the proof into three steps.
Step1: The sets $\{z : \widehat{v}(z) > \bar{v}(z), \widehat{v}(z) > (\widehat{u} - h)(z)\}$ and $\{z : \widehat{u}(z) > \bar{u}(z), \widehat{u}(z) > (\widehat{v} + g)(z)\}$ are exceptional.
To prove that any compact subset set of $\{z : \widehat{v}(z) > \bar{v}(z), \widehat{v}(z) > (\widehat{u} - h)(z)\}$ is exceptional, assume that there exists a compact non-exceptional subset F of $\{z : \widehat{v}(z) \geq \bar{v}(z) + \delta, \widehat{v}(z) \geq (\widehat{u} - h)(z) + \delta\}$ for some $\delta > 0$. For any cofinely open neighbourhood A of F, since $H_A\bar{v}_F = \widetilde{U}_1\left(\widehat{H}_A\mu_{\bar{v},F}\right) = \widetilde{U}_1\mu_{\bar{v},F} = \bar{v}_F$, $\bar{v}_F$ takes its

supremum on the cofine closure ${}^rF$ of F. Put $\gamma = \text{q.e.sup}\ \bar{v}_F = \text{q.e.sup}\ \widehat{v}_F$ and $F_\gamma = \{z : \widehat{v}_F(z) = \gamma\} \subset {}^rF$. Then any cofine neighbourhood of F_γ has positive capacity. Suppose that $\text{q.e.sup}(\bar{u} - h - \bar{v}_{F^c}) = \gamma$, then $\text{q.e.sup}(\widehat{u} - h - \widehat{v}_{F^c}) = \gamma$. Since $\widehat{u} - h - \widehat{v}_{F^c} \le \widehat{v}_F$, the q.e. supremum of $\widehat{u} - h - \widehat{v}_{F^c}$ is attained on $F_\gamma \subset {}^rF \subset F$. But, this is absurd, because $\widehat{u} - h - \widehat{v}_{F^c}$ is a cofinely continuous function dominated by $\widehat{v}_F - \delta$ on F. Therefore $\bar{u} - h - \bar{v}_{F^c} \le \eta$ q.e. for some $\eta < \gamma$. In particular $\bar{u} - h \le \bar{v}_F \wedge \eta + \bar{v}_{F^c}$. Since $\bar{v}_F \wedge \eta + v_{F^c}$ is an excessive function satisfying

$$g \le \bar{u} - \bar{v} \le \bar{u} - (v_F \wedge \eta + v_{F^c}) \le h,$$

this contradicts to the minimality of $\bar{v}$. Now we have shown that the set $\{z : \widehat{v}(z) > \bar{v}(z), \widehat{v}(z) > (\widehat{u} - h)(z)\}$ is exceptional. The exceptionality of $\{z : \widehat{u}(z) > \bar{u}(z), \widehat{v}(u) > (\widehat{v} + g)(z)\}$ follows similarly.

Step2: $\widehat{u} - \widehat{v} = \bar{u} - \bar{v}$ q.e.

If $(\widehat{u} - \bar{u})(z) < (\widehat{v} - \bar{v})(z)$, then $\widehat{v}(z) > \bar{v}(z)$ and $\widehat{v}(z) > (\widehat{u} - h)(z)$, because $(\widehat{u} - h)(z) < (\bar{u} + \widehat{v} - \bar{v} - h)(z) \le \widehat{v}(z)$. Hence $\{z : (\widehat{u} - \bar{u})(z) < (\widehat{v} - \bar{v})(z)\}$ is exceptional from step 1. Similarly $\{z : (\widehat{u} - \bar{u})(z) > (\widehat{v} - \bar{v})(z)\}$ is exceptional.

Step3: $\widehat{u} = \bar{u}$ and $\widehat{v} = \bar{v}$ q.e.

Note that $\mu_{\bar{u}}$ and $\mu_{\bar{v}}$ are mutually singular. In fact, if we can write $\mu_{\bar{u}} = \mu_{\bar{u}}^{(s)} + f \cdot \mu_{\bar{v}}$ for some non-negative measure $\mu_{\bar{u}}^{(s)}$ and non-negative function f such that $\langle \mu_{\bar{v}}, f \rangle > 0$, then

$$\bar{u} - \bar{v} = \widetilde{U}_1 \mu_{\bar{u}} - \widetilde{U}_1 \mu_{\bar{v}} = \widetilde{U}_1 \left(\mu_{\bar{u}}^{(s)} + (f - f \wedge 1) \cdot \mu_{\bar{v}} \right) - \widetilde{U}_1 \left((1 - f \wedge 1) \cdot \mu_{\bar{v}} \right)$$

which contradicts to the minimality of $\bar{u}$ and $\bar{v}$ in Theorem 2.2. Hence there exists a Borel set B such that $\mu_{\bar{u}}(\cdot) = \mu_{\bar{u}}(B \cap \cdot)$ and $\mu_{\bar{v}}(\cdot) = \mu_{\bar{v}}(B^c \cap \cdot)$. By virtue of Lemma 3.2, if $\{z : \widehat{v}(z) \neq \bar{v}(z)\}$ is not exceptional, then there exists a compact non-excdeptional set $F \subset \{z : \widehat{v}(z) > \bar{v}(z)\}$. Since $\widehat{v}_{F^c} = \bar{v}_{F^c}$ q.e. on F by Lemma 3.2, $\widehat{v}_F > \bar{v}_F$ q.e. on F and, in particular, $\mu_{\bar{v}}(F) = \mu_{\bar{v}}(B^c \cap F) > 0$. We may assume that $F \subset B^c$. Then $\mu_{\bar{u}}(F) = 0$ and hence $\widehat{u} = \bar{u}$ q.e. on F. In fact, if $\widehat{u} > \bar{u}$ on a non-exceptional set $K \subset F$, then $\widehat{u}_{K^c} = \bar{u}_{K^c}$ q.e. on K and hence $\widehat{u}_K > \bar{u}_K$ q.e. on K. This implies $\mu_{\bar{u}}(K) > 0$ which is impossible because $\mu_{\bar{u}}(F) = 0$. Therefore

$$(\widehat{u} - \widehat{v}) - (\bar{u} - \bar{v}) = -(\widehat{v} - \bar{v}) < 0$$

q.e. on F which contradicts to the assertion of step 2. $\square$

Since $\bar{u}$ is finely continuous, $\bar{u}(Z_t)$ is right continuous a.s. P_z for q.e.z. Similarly, $\widehat{u}(\widehat{Z}_t)$ is right continuous a.s. $\widehat{P}_z$ for q.e.z. Since $\bar{u} = \widehat{u}$, it becomes continuous along the sample paths. In fact, for any $f, g \ge 0$ and $t > 0$,

$$\begin{aligned}
&E_{f\cdot\nu}\left(g(Z_t) : \bar{u}(Z_s)_- \neq \bar{u}(Z_s), \exists s \in (0,t)\right) \\
&= \widehat{E}_{g\cdot\nu}\left(f(\widehat{Z}_t) : \bar{u}(\widehat{Z}_s)_+ \neq \bar{u}(\widehat{Z}_s), \exists s \in (0,t)\right) \\
&= \widehat{E}_{g\cdot\nu}\left(f(\widehat{Z}_t) : \widehat{u}(\widehat{Z}_s)_+ \neq \widehat{u}(\widehat{Z}_s), \exists s \in (0,t)\right) = 0.
\end{aligned}$$

The similar result also holds for $\bar{v}$. Hence

$$P_z\left(\bar{u}(Z_t) \text{ and } \bar{v}(Z_t) \text{ are continuous for } t > 0\right) = 1 \tag{3.5}$$

for a.e.z. By operating the transition function p_s and letting $s \to 0$, (3.5) holds for q.e.z.

Let $\mathcal{J} = \{u = u_1 - u_2 + w; u_i \in \mathcal{P}, w \in \mathcal{W}\}$. As in [[14];§5], $\mathcal{E}$ can be extended to $\mathcal{J} \times \mathcal{J}$ by $\mathcal{E}(u,v) = \lim_{\alpha\to\infty} \mathcal{E}(\alpha G_\alpha u, v)$ for $u, v \in \mathcal{J}$.

LEMMA 3.4. *The function $\bar{w} := \bar{u} - \bar{v}$ is the unique function of $\mathcal{J}$ such that, $\bar{w} = \widehat{w}$, $g \le \bar{w} \le h$ and, for any $w \in \mathcal{J}$ satisfying $g \le w \le h$,*

$$\mathcal{A}_1(\bar{w}, \bar{w}) \le \mathcal{E}_1(\bar{w}, w). \tag{3.6}$$

Proof. For any $w \in \mathcal{J}$ such that $g \le w$, since

$$\mathcal{E}_1(\bar{u}_n, \bar{u}_n - \bar{v} - w) = \frac{1}{\varepsilon_n}\left((\bar{u}_n - \bar{v} - g)^-, \bar{u}_n - \bar{v} - w\right) \le 0,$$

it holds that

$$\mathcal{A}_1(\bar{u}, \bar{u}) \le \varliminf_{n\to\infty} \mathcal{E}_1(\bar{u}_n.\bar{u}_n) \le \varliminf_{n\to\infty} \left(\mathcal{E}_1(\bar{u}_n, \bar{v}) + \mathcal{E}_1(\bar{u}_n, w)\right). \tag{3.7}$$

For any $p \in \mathcal{P}$, since $\mathcal{E}_1(\bar{u}_n, p) = -\mathcal{E}_1(p, \bar{u}_n) + 2\mathcal{A}_1(\bar{u}_n, p)$ and $\alpha G_{\alpha+1}\bar{u}_n$ is increasing relative to α and n, we have

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{E}_1(\bar{u}_n, p) &= -\lim_{n\to\infty}\lim_{\alpha\to\infty} \mathcal{E}_1(p, \alpha G_{\alpha+1}\bar{u}_n) + 2\mathcal{A}_1(\bar{u}, p)\\
&= -\lim_{\alpha\to\infty} \mathcal{E}_1(p, \alpha G_{\alpha+1}\bar{u}) + 2\lim_{\alpha\to\infty} \mathcal{A}_1(\alpha G_{\alpha+1}\bar{u}, p)\\
&= \lim_{\alpha\to\infty} \mathcal{E}_1(\alpha G_{\alpha+1}\bar{u}, p) = \mathcal{E}_1(\bar{u}, p).
\end{aligned}$$

This relation can be extended to all $p \in \mathcal{J}$ and hence, by (3.7),

$$\mathcal{A}_1(\bar{u}, \bar{u}) \le \mathcal{E}_1(\bar{u}, \bar{v}) + \mathcal{E}_1(\bar{u}, w).$$

Furthermore, since $\widehat{v} = \bar{v}$ q.e. from Lemma 3.3,

$$\begin{aligned}
\mathcal{E}_1(\bar{u}, \bar{v}) &= \lim_{\alpha\to\infty} \mathcal{E}_1(\alpha G_{\alpha+1}\bar{u}, \bar{v}) = \lim_{\alpha\to\infty} \mathcal{E}_1\left(\bar{u}, \alpha\widehat{G}_{\alpha+1}\bar{v}\right)\\
&= \int \widehat{v}\, d\mu_{\bar{u}} = \int \bar{v}\, d\mu_{\bar{u}}\\
&= \lim_{\alpha\to\infty} \mathcal{E}_1(\bar{u}, \alpha G_{\alpha+1}\bar{v}). \qquad (3.8)
\end{aligned}$$

Thus we get that

$$\mathcal{A}_1(\bar{u}, \bar{u}) \le \lim_{\alpha\to\infty} \mathcal{E}_1(\bar{u}, \alpha G_{\alpha+1}\bar{v}) + \mathcal{E}_1(\bar{u}, w).$$

Similarly, for any $w \in \mathcal{J}$ such that $w \le h$,

$$\mathcal{A}_1(\bar{v}, \bar{v}) \le \lim_{\alpha\to\infty} \mathcal{E}_1(\alpha G_{\alpha+1}\bar{v}, \bar{u}) - \mathcal{E}_1(\bar{v}, w).$$

Therefore, for any $w \in \mathcal{J}$ such that $g \le w \le h$,

$$\begin{aligned}
\mathcal{E}_1(\bar{u} - \bar{v}, w) &\ge \mathcal{A}_1(\bar{u}, \bar{u}) + \mathcal{A}_1(\bar{v}, \bar{v}) - \lim_{\alpha\to\infty}\{\mathcal{E}_1(\bar{u}, \alpha G_{\alpha+1}\bar{v}) + \mathcal{E}_1(\alpha G_{\alpha+1}\bar{v}, \bar{u})\}\\
&= \mathcal{A}_1(\bar{u}, \bar{u}) + \mathcal{A}_1(\bar{v}, \bar{v}) - 2\lim_{\alpha\to\infty} \mathcal{A}_1(\bar{u}, \alpha G_{\alpha+1}\bar{v})\\
&= \mathcal{A}_1(\bar{u} - \bar{v}, \bar{u} - \bar{v}),
\end{aligned}$$

that is, (3.6) holds. To prove the uniqueness of the solution, suppose that $w_1, w_2 \in \mathcal{J}$ satisfy the properties of the lemma. Since (3.8) holds for w_1 and w_2 instead of $\bar{u}$ and $\bar{v}$, respectively,

$$\begin{aligned}\mathcal{A}_1(w_1, w_2) + \mathcal{A}_1(w_2.w_1) &= \lim_{\alpha\to\infty} \left(\mathcal{A}_1(w_1, \alpha G_{\alpha+1} w_2) + \mathcal{A}_1(\alpha G_{\alpha+1} w_2, w_1)\right) \\ &= \lim_{\alpha\to\infty} \left(\mathcal{E}_1(w_1, \alpha G_{\alpha+1} w_2) + \mathcal{E}_1(\alpha G_{\alpha+1} w_2, w_1)\right) \\ &= \mathcal{E}_1(w_1, w_2) + \mathcal{E}_1(w_2, w_1).\end{aligned}$$

Hence, from (3.6),

$$\begin{aligned}\mathcal{A}_1(w_1 - w_2, w_1 - w_2) &= \mathcal{A}_1(w_1, w_1) + \mathcal{A}_1(w_2, w_2) - \mathcal{A}_1(w_1, w_2) - \mathcal{A}_1(w_2, w_1) \\ &= \mathcal{A}_1(w_1, w_1) + \mathcal{A}_1(w_2, w_2) - \mathcal{E}_1(w_1, w_2) - \mathcal{E}_1(w_2, w_1) \le 0\end{aligned}$$

which implies $w_1 = w_2$ a.e. □

Put $\dot{\sigma}_k = \dot{\sigma}_{C_k}$ and $\dot{\tau}_k = \dot{\sigma}_{D_k}$. Since C_k and D_k are decreasing, $\dot{\sigma} = \lim_{k\to\infty} \sigma_k$ and $\dot{\tau} = \lim_{k\to\infty} \tau_k$ exist as increasing limits. Clearly $\dot{\sigma} \le \dot{\sigma}_C$ and $\dot{\tau} \le \dot{\sigma}_D$.

Lemma 3.5. *For q.e. $z \in Z$, $\dot{\sigma}_C = \dot{\sigma}$ and $\dot{\sigma}_D = \dot{\tau}$ a.s. P_z.*

Proof. We shall only prove the assertion for $\dot{\sigma}_C$. For any $\ell \le k$,

$$\bar{u}_\ell(Z_{\dot{\sigma}_k}) \le \bar{u}_k(Z_{\dot{\sigma}_k}) \le (g + \bar{v})(Z_{\dot{\sigma}_k}).$$

Hence, by letting $k \uparrow \infty$ and then $\ell \uparrow \infty$, we get from (3.5) that $\bar{u}(Z_{\dot{\sigma}}) \le (g + \bar{v})(Z_{\dot{\sigma}})$. Hence $\dot{\sigma}_C \le \dot{\sigma}$. □

Since $\bar{u}$ and $\bar{v}$ are 1-excessive,

$$\bar{u}(z) \ge E_z\left(e^{-\sigma}\bar{u}(Z_\sigma)\right), \tag{3.9}$$

$$\bar{v}(z) \ge E_z\left(e^{-\tau}\bar{v}(Z_\tau)\right), \tag{3.10}$$

for any stopping times σ and τ. From the definition, $\bar{u}_k = \frac{1}{\varepsilon_k} R_1\left((\bar{u}_k - g - \bar{v})^-\right)$ q.e. Hence, for any stopping time σ such that $\sigma \le \dot{\sigma}_k$,

$$\begin{aligned}\bar{u}_k(z) &= \frac{1}{\varepsilon_k} E_z\left(\int_0^\infty e^{-t}(\bar{u}_k - g - \bar{v})^-(Z_t)dt\right) \\ &= \frac{1}{\varepsilon_k} E_z\left(\int_\sigma^\infty e^{-t}(\bar{u}_k - g - \bar{v})^-(Z_t)dt\right) \\ &= E_z\left(e^{-\sigma}\bar{u}_k(Z_\sigma)\right) \quad q.e.z.\end{aligned}\tag{3.11}$$

Similarly, if $\tau \le \dot{\tau}_k$, then

$$\bar{v}_k(z) = E_z\left(e^{-\tau}\bar{v}_k(Z_\tau)\right). \tag{3.12}$$

Theorem 3.6. *Suppose that g and h are quasi-continuous functions of $\mathcal{F}$ satisfying the separability condition. Then*

$$\bar{u}(z) - \bar{v}(z) = \sup_\sigma \inf_\tau J_z(\sigma, \tau) = \inf_\tau \sup_\sigma J_z(\sigma, \tau) \quad q.e. \tag{3.13}$$

Furthermore, $(\dot{\sigma}_C, \dot{\sigma}_D)$ is the saddle point of $J_z(\sigma, \tau)$ and $\bar{u} - \bar{v}$ is the unique solution of the quasi-variational inequality (3.6).

Proof. For any stopping time τ, applying (3.10) and (3.11) for $\dot\sigma_k\wedge\tau\le\dot\sigma_k$, we have

$$\begin{aligned}\bar u_k(z)-\bar v(z)&\le E_z\left(e^{-\dot\sigma_k\wedge\tau}(\bar u_k-\bar v)(Z_{\dot\sigma_k\wedge\tau})\right)\\&=E_z\left(e^{-\dot\sigma_k}(\bar u_k-\bar v)(Z_{\dot\sigma_k}):\dot\sigma_k\le\tau\right)+E_z\left(e^{-\tau}(\bar u_k-\bar v)(Z_\tau):\tau<\dot\sigma_k\right)\\&\le E_z\left(e^{-\dot\sigma_k}g(Z_{\dot\sigma_k}):\dot\sigma_k\le\tau\right)+E_z\left(e^{-\tau}h(Z_\tau):\tau<\dot\sigma_k\right).\end{aligned}$$

Then, by letting $k\to\infty$, we have

$$\bar u(z)-\bar v(z)\le E_z\left(e^{-\dot\sigma}g(Z_{\dot\sigma}):\dot\sigma\le\tau\right)+E_z\left(e^{-\tau}h(Z_\tau):\tau<\dot\sigma\right).$$

Similarly, for any stopping time σ, by considering $\sigma\wedge\dot\tau_k$, it holds that

$$E_z\left(e^{-\sigma}g(Z_\sigma):\sigma<\dot\tau\right)+E_z\left(e^{-\dot\tau}h(Z_{\dot\tau}):\dot\tau\le\sigma\right)\le\bar u(z)-\bar v(z).$$

Since $\dot\sigma=\dot\sigma_C$ and $\dot\tau=\dot\sigma_D$ from Lemma 3.5, the assertion of the theorem follows. ◻

(III) General two obstacles case:
In this case, we assume that g and h are quasi-continuous functions of $\mathcal{W}$. As in the preceding section, put $g^{(k)}=kR_kg$ and $h^{(k)}=kR_kh$. Then the separability condition holds for the obstacles $(g^{(k)},h^{(k)})$. For any stopping times σ and τ, put

$$J_z^{(k)}(\sigma,\tau)=E_z\left(e^{-\sigma\wedge\tau}\left(g^{(k)}(Z_\sigma)I_{\{\sigma\le\tau\}}+h^{(k)}(Z_\tau)I_{\{\tau<\sigma\}}\right)\right).\tag{3.14}$$

Let $(\bar u^{(k)},\bar v^{(k)})$ be the 1-excessive modifications of the minimal pair of functions determined by Theorem 2.2 for $(g^{(k)},h^{(k)})$ and put $\bar w^{(k)}=\bar u^{(k)}-\bar v^{(k)}$. Then $\widehat w^{(k)}=\bar w^{(k)}$ q.e. and satisfies

$$\bar w^{(k)}(z)=\sup_\sigma\inf_\tau J_z^{(k)}(\sigma,\tau)=\inf_\tau\sup_\sigma J_z^{(k)}(\sigma,\tau),\tag{3.15}$$

where $\widehat w^{(k)}=\lim_{n\to\infty}n\widehat R_n\bar w^{(k)}$. By virtue of (3.5), $\bar w^{(k)}$ is continuous along the sample paths almost surely.

THEOREM 3.7. *Suppose that g,h are quasi-continuous functions of $\mathcal{W}$. Then*

$$\bar w(z):=\sup_\sigma\inf_\tau J_z(\sigma,\tau)=\inf_\tau\sup_\sigma J_z(\sigma,\tau)\tag{3.16}$$

belongs to $\mathcal{F}$ and satisfies, for any $w\in\mathcal{W}$ such that $g\le w\le h$,

$$\mathcal{A}_1(\bar w,\bar w)\le\mathcal{E}_1(\bar w,w).\tag{3.17}$$

Moreover, the pair of the entry times $(\dot\sigma_C,\dot\sigma_D)$ is a saddle point of $J_z(\sigma,\tau)$.

Proof. For any stopping times σ, τ and q.e.z,

$$E_z\left(e^{-\sigma}|g^{(k)}-g|(Z_\sigma)I_{\{\sigma\le\tau\}}\right)\le E_z\left(e^{-\sigma}\widetilde e_{|g^{(k)}-g|}(Z_\sigma)\right)\le\widetilde e_{|g^{(k)}-g|}(z)\tag{3.18}$$

and

$$E_z\left(e^{-\tau}|h^{(k)}-h|(Z_\tau)I_{\{\tau<\sigma\}}\right)\le E_z\left(e^{-\tau}\widetilde e_{|h^{(k)}-h|}(Z_\tau)\right)\le\widetilde e_{|h^{(k)}-h|}(z).\tag{3.19}$$

Hence

$$|J_z^{(k)}(\sigma,\tau)-J_z(\sigma,\tau)|\le\widetilde e_{|g^{(k)}-g|}(z)+\widetilde e_{|h^{(k)}-h|}(z)\quad q.e.$$

By virtue of Lemmas 1.5 and 2.3, there exists an increasing sequence of closed sets $\{F_n\}$ and a subsequence $n_k \uparrow \infty$ such that $\mathrm{Cap}(Z \backslash F_n) \to 0$ and $\lim_{k\to\infty} \widetilde{e}_{|g^{(n_k)}-g|} = 0$, $\lim_{k\to\infty} \widetilde{e}_{|h^{(n_k)}-h|} = 0$ in $\mathcal{F}$ and uniformly on each set F_n. Hence

$$\lim_{k\to\infty} \inf_\tau \sup_\sigma J_z^{(n_k)}(\sigma,\tau) = \inf_\tau \sup_\sigma J_z(\sigma,\tau) \quad \text{on } F_n.$$

(3.15) combined with (3.18) and (3.19) implies that

$$|\bar{w}^{(k)} - \bar{w}^{(l)}| \le \widetilde{e}_{|g^{(k)}-g|} + \widetilde{e}_{|g^{(l)}-g|} + \widetilde{e}_{|h^{(k)}-h|} + \widetilde{e}_{|h^{(l)}-h|} \quad q.e.$$

Therefore $\lim_{k\to\infty} \bar{w}^{(k)} = \bar{w}^{(\infty)}$ exists in $\mathcal{H}$ and a subsequence converges quasi-uniformly on each F_n and satisfies

$$\bar{w}^{(\infty)} = \sup_\sigma \inf_\tau J_z(\sigma,\tau) = \inf_\tau \sup_\sigma J_z(\sigma,\tau) \quad q.e. z \in F_n.$$

Letting $n \to \infty$, we get that $\bar{w}^{(\infty)} = \bar{w}$ q.e. By virtue of the quasi-variational inequality (3.6) applied for $\bar{w}^{(k)}$,

$$\begin{aligned}\mathcal{A}_1(\bar{w}^{(k)}, \bar{w}^{(k)}) &\le \mathcal{E}_1(\bar{w}^{(k)}, g^{(k)}) = \mathcal{E}_1(k\widehat{G}_k \bar{w}^{(k)}, g) \\ &\le \|g\|_{\mathcal{W}} \|k\widehat{G}_k \bar{w}^{(k)}\|_{\mathcal{F}} \le C_1 \|g\|_{\mathcal{W}} \|\bar{w}^{(k)}\|_{\mathcal{F}}.\end{aligned}$$

Hence $\|\bar{w}^{(k)}\|_{\mathcal{F}}$ and $\|k\widehat{G}_k \bar{w}^{(k)}\|_{\mathcal{F}}$ are uniformly bounded relative to k. This combined with $\lim_{k\to\infty} \bar{w}^{(k)} = \lim_{k\to\infty} k\widehat{G}_k \bar{w}^{(k)} = \bar{w}$ in $\mathcal{H}$, implies that $\lim_{k\to\infty} \bar{w}^{(k)} = \lim_{k\to\infty} k\widehat{G}_k \bar{w}^{(k)} = \bar{w}$ weakly in $\mathcal{F}$. Therefore, for any $w \in \mathcal{W}$ such that $g \le w \le h$,

$$\mathcal{A}_1(\bar{w}^{(k)}, \bar{w}^{(k)}) \le \mathcal{E}_1(\bar{w}^{(k)}, kG_k w) = \mathcal{E}_1(k\widehat{G}_k \bar{w}^{(k)}, w).$$

By letting $k \to \infty$, the quasi-variational inequality (3.17) follows.
The proof of the last part of the theorem is similar to [19]. It suffices to prove the following inequalities for any stopping times σ and τ such that $\tau \le \dot{\sigma}_C$ and $\sigma \le \dot{\sigma}_D$.

$$\bar{w}(z) \le E_z\left(e^{-\tau} \bar{w}(Z_\tau)\right) \quad q.e. \tag{3.20}$$

$$\bar{w}(z) \ge E_z\left(e^{-\sigma} \bar{w}(Z_\sigma)\right) \quad q.e. \tag{3.21}$$

In fact, if these hold, by noting $\bar{w}(Z_{\dot{\sigma}_C}) = g(Z_{\dot{\sigma}_C})$, we have for any stopping time τ,

$$\begin{aligned}J_z(\dot{\sigma}_C, \tau) &= E_z\left(e^{-\dot{\sigma}_C} g(Z_{\dot{\sigma}_C}) I_{\{\dot{\sigma}_C \le \tau\}} + e^{-\tau} h(Z_\tau) I_{\{\tau < \dot{\sigma}_C\}}\right) \\ &\ge E_z\left(e^{-\dot{\sigma}_C} \bar{w}(Z_{\dot{\sigma}_C}) I_{\{\dot{\sigma}_C \le \tau\}} + e^{-\tau} \bar{w}(Z_\tau) I_{\{\tau < \dot{\sigma}_C\}}\right) \\ &= E_z\left(e^{-\dot{\sigma}_C \wedge \tau} \left(\bar{w}(Z_{\dot{\sigma}_C \wedge \tau}\right)\right) \ge \bar{w}(z).\end{aligned}$$

Similarly, $J_z(\sigma, \dot{\sigma}_D) \le \bar{w}$ q.e. for any stopping time σ.
Let $C^{(k)} = \{z : \bar{w}^{(k)}(z) \le g^{(k)}(z)\}$, $D^{(k)} = \{z : h^{(k)}(z) \le \bar{w}^{(k)}(z)\}$, $\dot{\sigma}^{(k)} = \dot{\sigma}_{C^{(k)}}$ and $\dot{\tau}^{(k)} = \dot{\sigma}_{D^{(k)}}$. Further, for each positive number γ, let $\eta_\gamma = \inf\{t \ge 0 : \bar{w}(Z_t) + \gamma \ge h(Z_t)\}$. If $t < \eta_\gamma(\omega)$, then $\bar{w}(Z_t(\omega)) + \gamma < h(Z_t(\omega))$. Noting that $\lim_{k\to\infty} \bar{w}^{(k)} = \bar{w}$ and $\lim_{k\to\infty} h^{(k)} = h$ uniformly on each F_n, we can find k_0 such that $|\bar{w}^{(k)} - \bar{w}| \le \gamma/2$ and $|h^{(k)} - h| < \gamma/2$ on F_n for any $k \ge k_0$. Therefore, $t < \dot{\tau}^{(k)}$ and hence $\eta_\gamma \wedge \dot{\sigma}_{F_n} \le \dot{\tau}^{(k)}$. In view of (3.9) and (3.12), since (3.21) holds for $\bar{w}^{(k)}$ and $\dot{\tau}^{(k)}$ instead of $\bar{w}$ and $\dot{\sigma}_D$, respectively, it then holds that

$$\bar{w}^{(k)}(z) \ge E_z\left(e^{-(\eta_\gamma \wedge \tau_{F_n} \wedge \sigma)} w^{(k)}(Z_{\eta_\gamma \wedge \tau_{F_n} \wedge \sigma})\right) \quad q.e.$$

Letting $k \to \infty$ and then $n \to \infty$, we have

$$\bar{w}(z) \geq E_z \left(e^{-(\eta_\gamma \wedge \sigma)} \bar{w}(Z_{\eta_\gamma \wedge \sigma}) \right) \quad q.e.$$

Since $\bar{w}(Z_{\eta_\gamma}) + \gamma \geq h(Z_{\eta_\gamma})$ and η_γ is increasing as $\gamma \downarrow 0$, $\widehat{w}(Z_{\eta_0}) = \bar{w}(Z_{\eta_0}) \geq h(Z_{\eta_0})$ for $\eta_0 = \lim_{\gamma \to 0} \eta_\gamma$. This yields that $\dot{\sigma}_D \leq \eta_0$ and hence

$$\bar{w}(z) \geq E_z \left(e^{-\sigma} \bar{w}(Z_\sigma) \right)$$

for any stopping time σ such that $\sigma \leq \dot{\sigma}_D$. The proof of (3.20) is similar. □

Example Let (Z_t, P_z) be the space-time diffusion process corresponding to the generator $A^{(t)}\varphi(x) = \frac{d^2\varphi}{dx^2}$ for $t < 1$ and $A^{(t)}\varphi(x) = \frac{1}{2}\frac{d^2\varphi}{dx^2}$ for $t \geq 1$. Denote by q_t and K_α the transition function and resolvent of 1-dimensional Brownian motion, respectively. Taking a non-negative, non-zero continuous function ϕ on R^1 with compact support, let $\varphi(x) = K_2\phi(x)$,

$$g(s,x) = \begin{cases} e^{-(3-2s)} q_{2(1-s)}\varphi(x), & s < 1 \\ e^{-s}\varphi(x), & s \geq 1 \end{cases}$$

and

$$h(s,x) = \begin{cases} 2e^{-(3-s)/2} q_{2(1-s)}\varphi(x), & s < 1 \\ 2e^{-s}\varphi(x), & s \geq 1. \end{cases}$$

Furthermore let $\bar{v} = 0$ and

$$\bar{u}(s,x) = \begin{cases} e^{-(2-s)} q_{2(1-s)}\varphi(x), & s < 1 \\ e^{-s}\varphi(x), & s \geq 1. \end{cases}$$

Then $g \leq \bar{u} - \bar{v} \leq h$. Since $\bar{u}(s,x) = E_{(s,x)}\left(\int_0^\infty e^{-t}\xi \otimes \phi(Z_t)dt\right)$, for $\xi(t) = e^{-t}I_{\{t \geq 1\}}$, $\bar{u}$ is 1-excessive function of Z_t and satisfies

$$\mathcal{A}_1(\bar{u}, \bar{u}) = \frac{1}{2}e^{-2}\int_{-\infty}^{\infty} \varphi(x)\phi(x)dx.$$

Further, for any $w \in \mathcal{J}$ such that $g \leq w \leq h$,

$$\mathcal{E}_1(\bar{u}, w) = \int_1^\infty \int_{-\infty}^\infty e^{-s}\phi(x)w(s,x)dxds \geq \int_1^\infty \int_{-\infty}^\infty e^{-s}\phi(x)g(s,x)dxds = \mathcal{A}_1(\bar{u}, \bar{u}).$$

Therefore $\bar{u} - \bar{v} = \bar{u}$ is a solution of (3.6) and the saddle point $(\dot{\sigma}_C, \dot{\sigma}_D)$ of $J_z(\sigma, \tau)$ in Theorem 3.6 is given by $\dot{\sigma}_C = (1 - \tau(0)) \vee 0$ and $\dot{\sigma}_D = \infty$.

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